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Three-point function of semiclassical states at weak coupling

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We give the derivation of the previously announced analytic expression for the correlation function of three heavy non-BPS operators in $\mathcal{N} = 4$ super-Yang-Mills theory at weak coupling. The three operators belong to three different $su(2)$ sectors and are dual to three classical strings moving on the sphere. Our computation is based on the reformulation of the problem in terms of the Bethe Ansatz for periodic XXX spin-1/2 chains. In these terms the three operators are described by long-wave-length excitations over the ferromagnetic vacuum, for which the number of the overturned spins is a finite fraction of the length of the chain, and the classical limit is known as the Sutherland limit. Technically our main result is a factorized operator expression for the scalar product of two Bethe states. The derivation is based on a fermionic representation of Slavnov's determinant formula, and a subsequent bosonisation.

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1 Introduction

During the last decade, starting with the pioneer paper by Minahan and Zarembo [1], a vast integrable structure has been unveiled in the $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory and in its dual supersymmetric string theory in the $AdS_5 \times S^5$ space-time. The integrable structure, known by the name of Integrability, was formulated in terms of an effective long range spin chain having $PSL(2, 2|4)$ symmetry. For a recent review see [2].

The spectral problem, i.e. the problem of computing the anomalous dimensions of the gauge-invariant states in SYM and their string counterparts, is nowadays considered as conceptually solved. Spectacular progress was achieved in the last years in the computation of the gluon amplitudes [3, 4] and Wilson loops [5, 6]. The next step towards the complete solution is to compute the three-point function of gauge-invariant operators representing traces of products of fundamental fields. This is obviously an extremely hard problem, but the encouraging developments over the last several years [7–11] raise the hope that Integrability can be used for this problem as well.

Of special interest are the correlation functions of one-trace operators in the classical limit when the length of the traces is very large. Such operators are dual to extended classical strings in the $\mathrm{AdS}_5 \times \mathrm{S}^5$ background. This correspondence allows to approach the problem both at strong and at weak coupling. The classical, or long-trace, operators are described in terms of algebraic curves, which appear as finite-gap solutions of the Bethe equations in the classical limit, or as classical solutions of the string sigma model [12–14] (see also the review [15]).

On the string theory side, the problem of computing the correlation function of operators dual to classical spinning string solutions was addressed by several authors [16–22]. The three-point function of such operators can be thought of as a classical tunneling amplitude. To compute it, one should find the appropriate Euclidean classical solution for a world sheet embedded in $AdS_5 \times S^5$ and having the topology of a sphere with three punctures. However, there is not yet a consensus among the active workers on the field about the criteria to distinguish the relevant classical solution, neither there is an unambiguous prescription about how to construct the vertex operators associated with the punctures. The only case when the complete answer is known is that of two heavy and one light operators [17, 23, 24].

Alternatively, one can start with three one-trace operators in the weakly coupled gauge theory, where the computation of the correlation functions is a well defined problem. At tree level it is sufficient to count all possible planar sets of Wick contractions between the three operators. It was pointed out by Roiban and Volovich [8] that the calculation of correlation functions of gauge invariant operators reduces at the level of the spin chain to the calculation of the scalar products of states constructed out of B and C operators in the Algebraic Bethe Ansatz [25]. A systematic study of the case when the three operators belong to three different $su(2)$ sectors of the theory was presented by Escobedo, Gromov, Sever and Vieira [9]. The tree-level correlation function of three $su(2)$ operators was expressed in [9] in terms of scalar products of Bethe states in a periodic XXX spin chain with spin 1/2. The formalism developed in [9] was later applied to compute the correlation function of two heavy and one (or more) light operators [10, 26], and the comparison with the strong coupling results on the string theory side [17, 23, 24] showed a precise match. The limit of three heavy, or classical, operators was then obtained in [11] in the case when one of the operators is BPS type, that is, protected by the supersymmetry. The main result of [11] is an elegant analytic formula in the form of a contour integral. The derivation was based on the expansion formula for the scalar products of two generic Bethe states due to Korepin [27].

In this paper we tackle the general case when neither of the three heavy operators is BPS type. We compute, at tree-level, the correlation function of three non-protected classical operators belonging to

three different $su(2)$ sectors of the gauge theory. The principal object to be computed is the restricted scalar product of two Bethe states, in which part of the rapidities are frozen to a special value. Our result, which is a generalization of the main result of [11], was announced in the short note [28]. The analytic expression found in [28] is based on a factorization formula, which follows from the representation of the structure constant in terms of Slavnov-like determinants [29], proposed recently by Foda [30–32].

The plan of the paper is as follows. In Section 2 we review the basic notions about the XXX spin chain we are going to use, including the Gaudin norm, the Slavnov determinant formula for the scalar product, and its restricted version. Following Foda [30], we will perform the computations for the inhomogeneous spin chain, characterized by a set of external parameters (impurities) associated with the sites of the chain. Such a deformation of the problem allows to avoid ambiguous expressions containing poles and zeroes on the top of each other. In Section 3 we derive a representation of the Slavnov determinant in terms of free chiral fermions, and then perform bosonization. As a side result, we obtain a novel and potentially useful representation of the Izergin determinant for the domain wall partition function (DWPF) of the six-vertex model. In the limit when the number of Bethe roots tends to infinity, the bosonized expression decomposes into two computable factors, for which we find the analytic expression in Section 4. In section 5 we derive the structure constant of three classical non-BPS operators in the $sl(2)$ sector of $\mathcal{N} = 4$ SYM.

2 Inner product of Bethe states in the inhomogeneous XXX chain

2.1 The monodromy matrix

The local fluctuation variable in a XXX spin 1/2 chain can be in two states, $\downarrow$ and $\uparrow$, which can be thought of as a basis of a two-dimensional linear space V on $\mathbb{C}$. The spin chain is characterized by an isotropic Hamiltonian

$$H_{\text{XXX}} = -\sum_{m=1}^{L} \left(\sigma^+_m \sigma^-_{m+1} + \sigma^-_m \sigma^+_{m+1} + \tfrac{1}{2} \sigma^x_m \sigma^x_{m+1} \right) . \tag{2.1}$$

We will assume twisted periodic boundary conditions,

$$\sigma^{\pm}_{m+L} = \kappa^{\pm}\, \sigma^{\pm}_m \,, \tag{2.2}$$

which do not spoil the integrability, but allow to have a better control over the singularities. If the twist

$$\kappa \equiv \kappa_- / \kappa_+ = e^{i\phi} \tag{2.3}$$

is a pure phase, the twisted boundary conditions can be thought of as the effect of turning on a magnetic flux of strength ϕ [33, 34]. For us κ will be an unrestricted complex parameter.

In the framework of the Algebraic Bethe Ansatz [25], the spin chain is characterized by an R-matrix $R_{12}(u, v)$ acting in the tensor product $V_1 \otimes V_2$ of two copies of the target space,

$$R_{12}(u, v) = u - v + iP_{12} \,, \tag{2.4}$$

where P_{12} is the permutation operator [35]. We will consider the *inhomogeneous* XXX spin chain, characterized by background parameters (impurities) $\boldsymbol{\theta} = \{\theta_1, \ldots, \theta_L\}$ associated with the L sites

of the chain. The twisted monodromy matrix $\mathcal{T}_a(u) \in \mathrm{End}(V_a)$ is defined as the product of the R-matrices along the spin chain and a twist matrix $K = \left(\begin{smallmatrix} \kappa_+ & 0 \\ 0 & \kappa_- \end{smallmatrix}\right)$,

$$\mathcal{T}_a(u) \equiv K R_{1a}(u, \theta_1 + \tfrac{1}{2}i)\, R_{2a}(u, \theta_2 + \tfrac{1}{2}i) \dots R_{La}(u, \theta_L + \tfrac{1}{2}i) = K \begin{pmatrix} \mathcal{A}(u) & \mathcal{B}(u) \\ \mathcal{C}(u) & \mathcal{D}(u) \end{pmatrix}. \tag{2.5}$$

For the homogeneous XXX spin chain all θ_m are equal to 0. The advantage of introducing the twist and the inhomogeneity parameters is that the expressions for some scalar products, which are ambiguous for $\theta_m = 0$ and $\kappa = 1$, becomes well defined for generic θ_m and κ.

The matrix elements $\mathcal{A}, \mathcal{B}, \mathcal{C}, \mathcal{D}$ are operators in the Hilbert space $V = V_1 \otimes \dots V_L$ of the spin chain. The commutation relations between the elements of the monodromy matrix are determined by the relation

$$R_{12}(u-v)\mathcal{T}_1(u)\mathcal{T}_2(v) = \mathcal{T}_2(v)\mathcal{T}_1(u)\, R_{12}(u-v) \tag{2.6}$$

which follows from the Yang-Baxter equation for R. As a consequence of (2.6), the operators $\mathcal{B}(u)$, $\mathcal{C}(u)$, and the transfer matrices

$$\mathcal{T}(u) \equiv \mathrm{Tr}_a[\mathcal{T}_a(u)] = \mathcal{A}(u) + \mathcal{D}(u), \tag{2.7}$$

form commuting families:

$$[\mathcal{B}(u), \mathcal{B}(v)] = [\mathcal{C}(u), \mathcal{C}(v)] = [\mathcal{T}(u), \mathcal{T}(v)] = 0 \quad \text{for } u, v \in \mathbb{C}. \tag{2.8}$$

2.2 The Hilbert space as a Fock space for the pseudo-particles (the magnons)

One can give the Hilbert space V a structure of a Fock space generated by the action of the operators $\mathcal{B}(u)$ on the pseudo-vacuum $\langle \uparrow^L | = \left| \uparrow\uparrow \overset{L}{\dots} \uparrow \right\rangle$. The pseudo-vacuum is an eigenvector for the diagonal elements $\mathcal{A}$ and $\mathcal{D}$ and is annihilated by $\mathcal{C}$:

$$\mathcal{A}(u)\, |\uparrow^L\rangle = a(u)\, |\uparrow^L\rangle, \quad \mathcal{D}(u)|\uparrow^L\rangle = d(u)|\uparrow^L\rangle, \quad \mathcal{C}(u)|\uparrow^L\rangle = 0. \tag{2.9}$$

The dual Bethe states are generated by the action of the $\mathcal{C}$-operators on the dual pseudo-vacuum $\langle \uparrow^L | = \left\langle \uparrow\uparrow \overset{L}{\dots} \uparrow \right|$, which has the properties

$$\langle \uparrow^L |\, \mathcal{A}(u) = a(u)\, \langle \uparrow^L |, \quad \langle 0 | \mathcal{D}(u) = d(u)\, \langle \uparrow^L |, \quad \langle \uparrow^L | \mathcal{B}(u) = 0. \tag{2.10}$$

The functions $a(u)$ and $d(u)$ depend on the representation of the algebra (2.6), while the R-matrix (2.4) is universal. For the inhomogeneous XXX spin 1/2 magnet the functions $a(u)$ and $d(u)$ are given, according to (2.5), by

$$\begin{aligned} a(u) &= \kappa_+ \prod_{m=1}^{L} (u - \theta_m + i/2)\,, \\ d(u) &= \kappa_- \prod_{m=1}^{L} (u - \theta_m - i/2)\,. \end{aligned} \tag{2.11}$$

The algebraic construction of the Hilbert space does not use the particular form of the functions $a(u)$ and $d(u)$, which can be considered as free functional parameters [27].[1] The space of states is a

[1]In this case one speaks of generalized $SU(2)$ model.

closure of the linear span of all vectors of the form

$$\left|\mathbf{u}\right\rangle = \prod_{j=1}^{N} \mathcal{B}(u_j)\,|\!\uparrow^L\rangle. \tag{2.12}$$

Here and below we will use the shorthand notation [2]

$$\mathbf{u} = \mathbf{u}_N = \{u_1, \dots u_N\}. \tag{2.13}$$

The operator $\mathcal{B}(u)$ can be viewed as a creation operator of a pseudo-particle (magnon) with momentum $p = \log \frac{u+i/2}{u-i/2}$. Similarly, the dual space of states is a closure of the linear span of all vectors of the form[3]

$$\left\langle\mathbf{u}\right| = \langle\uparrow^L\!| \prod_{j=1}^{N} \mathcal{C}(u_j). \tag{2.14}$$

Such states are generic, or off-shell, Bethe states. A Bethe state (2.12) which is also an eigenstate of the transfer matrix (2.7) is called on-shell state. For a chain of length L there are 2^L linearly independent on-shell states. Applying repeatedly the RTT relations (2.6) for the upper triangular elements of the monodromy matrix, namely

$$\begin{aligned} \mathcal{A}(v)\mathcal{B}(u) &= \tfrac{u-v+i}{u-v}\,\mathcal{B}(u)\mathcal{A}(v) - \tfrac{i}{u-v}\mathcal{B}(v)\mathcal{A}(u), \\ \mathcal{D}(v)\mathcal{B}(u) &= \tfrac{u-v-i}{u-v}\,\mathcal{B}(u)\mathcal{A}(v) + \tfrac{i}{u-v}\mathcal{B}(v)\mathcal{D}(u), \end{aligned} \tag{2.15}$$

one obtains [35] the on-shell condition for the rapidities $\mathbf{u} = \{u_1, \dots u_N\}$, which coincide with the (twisted) Bethe equations

$$\frac{d(u_j)}{a(u_j)} \prod_{k=1}^{N} \frac{u_j - u_k + i}{u_j - u_k - i} = -1, \quad a = 1, \dots, N. \tag{2.16}$$

Since the XXX Hamiltonian is hermitian, the set of roots $\mathbf{u}$ of a Bethe eigenstate must be invariant under complex conjugation. The corresponding eigenvalue $T_{\mathbf{u}}(u)$ of the transfer matrix is

$$T_{\mathbf{u}}(u) = a(u)\frac{Q_{\mathbf{u}}(u-i)}{Q_{\mathbf{u}}(u)} + d(u)\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u)}, \tag{2.17}$$

where $Q_{\mathbf{u}}(u)$ is the Baxter polynomial

$$Q_{\mathbf{u}}(u) \equiv \prod_{j=1}^{N}(u - u_j). \tag{2.18}$$

The Bethe equations (2.16) can be considered as saddle-point equations for the Yang-Yang functional [25, 36], which we denote by $\mathscr{Y}_{\mathbf{u}}$,

$$\partial_{u_j}\mathscr{Y}_{\mathbf{u}} = 2\pi i n_j \quad (a = 1, \dots, N;\ n_j \in \mathbb{Z}). \tag{2.19}$$

[2] We will indicate the cardinality N only if this is required by the context.

[3] With the convention $\mathcal{B}(u)^\dagger = -\mathcal{C}(u^*)$, the state dual to $|\mathbf{v}\rangle$ is $(-1)^N\langle\mathbf{v}^*|$.

For the twisted periodic $\mathrm{XXX}_{1/2}$ spin chain, the Yang-Yang functional is given by

$$\begin{aligned}\mathscr{Y}_{\mathbf{u}} &= \sum_{j=1}^{N}\sum_{m=1}^{L}(u_j-\theta_m+\tfrac{1}{2}i)\log(u_j-\theta_m+\tfrac{1}{2}i)-(u_j-\theta_m-\tfrac{1}{2}i)\log(u_j-\theta_m-\tfrac{1}{2}i)\\ &+ \sum_{j<k}[(u_j-u_k+i)\log(u_j-u_k+i)-(u_j-u_k-i)\log(u_j-u_k-i)] \qquad (2.20)\\ &+ i\phi\sum_{j=1}^{N}u_j. \qquad (2.21)\end{aligned}$$

2.3 The pseudo-momentum

For each set of points $\mathbf{u}=\{u_j\}_{j=1}^N$, define the function $p_{\mathbf{u}}(u)$, called *pseudomomentum*[4]

$$e^{2ip_{\mathbf{u}}(u)} \equiv \frac{d(u)}{a(u)}\,\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u-i)} = \kappa\,\frac{Q_{\boldsymbol{\theta}}(u-i/2)}{Q_{\boldsymbol{\theta}}(u+i/2)}\,\frac{Q_{\mathbf{u}}(u+i)}{Q_{\mathbf{u}}(u-i)}. \qquad (2.22)$$

Here and below we will use the notation (2.18) for products over rapidities. In terms of the pseudo-momentum, the Bethe equations (2.16) read

$$e^{2ip_{\mathbf{u}}(u)} = -1, \quad u\in\mathbf{u}. \qquad (2.23)$$

The pseudo-momentum is determined by (2.22) modulo $i\pi$. To characterize this function completely, it is necessary to specify a set of integers (mode numbers) $\{n_j\}_{j=1}^N$, not necessarily different, so that $p(u_j)=\pi n_j$.

2.4 Slavnov's determinant formula for the inner product

The scalar product of two generic Bethe states can be computed from the algebra (2.6), the relations (2.9) and (2.10), and the convention $\langle\uparrow^L\mid\uparrow^L\rangle=1$. In general, the scalar product $\langle\mathbf{v}|\mathbf{u}\rangle$ of two Bethe states is given by a double sum over partitions of the sets $\mathbf{u}=\mathbf{u}_{_N}$ and $\mathbf{v}=\mathbf{v}_{_N}$, which becomes increasingly difficult to tackle when number of magnons N becomes large. A significant simplification occurs when one of the two sets of rapidities, say $\mathbf{u}$, satisfies the Bethe equations (2.16). It was discovered by N. Slavnov [29] that in this case the scalar product is a determinant. This is true for all integrable models with $A_1^{(1)}$ type R-matrix.

Let the set $\mathbf{u}$ satisfy the Bethe equations (2.16). Then $|\mathbf{u}\rangle$ is an eigenvector for the transfer matrix with eigenvalue $T_{\mathbf{u}}(u)$, given by eq. (2.17). It was shown by Slavnov [29, 37] that the scalar product with a generic Bethe state $\langle\mathbf{v}|$ is proportional to the determinant of the matrix of the derivatives of $T_{\mathbf{u}}(u)$, evaluated at the points of $\mathbf{v}$,

$$\langle\mathbf{v}|\mathbf{u}\rangle = \prod_{j=1}^{N}a(v_j)d(u_j)\,\mathscr{S}_{\mathbf{u},\mathbf{v}}\,; \qquad (2.24)$$

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} \overset{\text{def}}{=} \frac{1}{\prod_{j=1}^N a(v_j)}\,\frac{\det_{jk}\frac{\partial}{\partial u_j}T_{\mathbf{u}}(v_k)}{\det_{jk}\frac{1}{u_j-v_k}}. \qquad (2.25)$$

[4]There is no complete consensus about the terminology. The quantity we refer to as pseudo-momentum is related to the *counting function* $Z(u)$ by $e^{2ip_{\mathbf{u}}(u)}=(-1)^{N+L}e^{-iZ(u)}$.

(Eqs. (2.24)-(2.25) are equivalent to eq. (6.16) of [37].) The explicit expression for the matrix of the derivatives $\partial_{u_j} T_{\mathbf{u}}(v_k)$ is

$$\begin{aligned} -\,\partial T_{\mathbf{u}}(v_k)/\partial u_j &= t(u_j - v_k)\, a(v_k) \frac{Q_{\mathbf{u}}(v_k - i)}{Q_{\mathbf{u}}(v_k)} - t(v_k - u_j)\, d(v_k) \frac{Q_{\mathbf{u}}(v_k + i)}{Q_{\mathbf{u}}(v_k)} \\ &= a(v_k) \frac{Q_{\mathbf{u}}(v_k - i)}{Q_{\mathbf{u}}(v_k)}\, \Omega(u_j, v_k), \end{aligned} \tag{2.26}$$

where the kernel $\Omega(u,v)$ is defined by

$$\Omega(u,v) = t(u-v) - e^{2ip_{\mathbf{u}}(v)}\, t(v-u)\,, \qquad t(u) = \frac{1}{u} - \frac{1}{u+i}\,. \tag{2.27}$$

One can simplify the Slavnov determinantt (2.25) by noticing that $\prod_{k=1}^{N} \frac{Q_{\mathbf{u}}(v_k - i)}{Q_{\mathbf{u}}(v_k)} = \frac{\det_{jk} (u_j - v_k)^{-1}}{\det_{jk} (u_j - v_k + i)^{-1}}$. The resulting expression is

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = \frac{\det_{jk} \Omega(u_j, v_k)}{\det_{jk} \frac{1}{u_j - v_k + i}}. \tag{2.28}$$

In order to get rid of the factors $a(v_j)\, d(u_j)$ in (2.24), we rescale the annihilation/creation operators as

$$\mathcal{C}(u) \to \mathbb{C}(u) = \frac{\mathcal{C}(u)}{a(u)}, \quad \mathcal{B}(u) \to \mathbb{B}(u) = \frac{\mathcal{B}(u)}{d(u)} \tag{2.29}$$

and denote the rescaled inner product by $\langle \mathbf{u} | \mathbf{v} \rangle$:

$$\langle \mathbf{v} | \mathbf{u} \rangle \stackrel{\text{def}}{=} \langle \uparrow^L | \prod_{v \in \mathbf{v}} \mathbb{C}(u) \prod_{u \in \mathbf{u}} \mathbb{B}(u) | \uparrow^L \rangle = \mathscr{S}_{\mathbf{u},\mathbf{v}}\,. \tag{2.30}$$

Depending on the context, sometimes we will indicate the set of the inhomogeneity parameters and/or the cardinalities of the sets $\mathbf{u}, \mathbf{v}$ and $\boldsymbol{\theta}$,

$$\langle \mathbf{v} | \mathbf{u} \rangle = \langle \mathbf{v} | \mathbf{u} \rangle_{\boldsymbol{\theta}} = \langle \mathbf{v}_N | \mathbf{u}_N \rangle_{\boldsymbol{\theta}_L}. \tag{2.31}$$

2.5 The Gaudin norm

The hermitian conjugation compatible with the scalar product (2.24) is $\mathcal{C}(u) = -\mathcal{B}(\bar{u})^\dagger$, or

$$\mathbb{C}(u) = \frac{d(u)}{a(u)}\, \mathbb{B}(\bar{u})^\dagger. \tag{2.32}$$

Assuming that the sets $\mathbf{u}$ and $\mathbf{v}$ are invariant under complex conjugation, $\bar{\mathbf{u}} = \mathbf{u}$, $\bar{\mathbf{v}} = \mathbf{v}$, and taking the limit $\mathbf{v} \to \mathbf{u}$ in (2.28), one reproduces the determinant expression for the square of the norm of a Bethe eigenstate conjectured by Gaudin [38, 39] and proved by Korepin in [27]. The square of the norm is proportional to the Hessian of the Yang-Yang functional. In our normalization

$$\langle \mathbf{u} | \mathbf{u} \rangle = \frac{\det_{jk} \left(i \frac{\partial^2 \mathcal{Y}_{\mathbf{u}}}{\partial u_j \partial u_k} \right)}{\det_{jk} \frac{1}{1 + i(u_j - u_k)}}. \tag{2.33}$$

The explicit expression for the matrix of the second derivatives is

$$i \frac{\partial^2 \mathcal{Y}_{\mathbf{u}}}{\partial u_j \partial u_k} = \frac{2}{(u_j - u_k)^2 + 1} - \delta_{jk} \left(\sum_{l=1}^{N} \frac{2}{(u_j - u_l)^2 + 1} - \sum_{j=1}^{L} \frac{1}{(u_j - \theta_m)^2 + \frac14} \right). \tag{2.34}$$

3 Operator factorization formula for the inner product

3.1 Slavnov's determinant as a fermionic Fock space expectation value

The Slavnov determinant expression (2.28) for the scalar product can be formulated in terms of free fermions and represents a tau-function of the KP/Toda hierarchy [40–42]. For the Izergin determinant this was shown and used in [31,43,44], see also the review paper [45]. The two sets of Toda times are related to the moments of the two sets of rapidities, $\mathbf{u} = \{u_1, \ldots, u_N\}$ and $\mathbf{v} = \{v_1, \ldots, v_N\}$.

The fermion representation we are going to use here is not the most natural one from the point of view of integrable hierarchies, but it reveals a hidden factorization property of the scalar product, which can be used to find an analytic expression in the thermodynamical limit $N, L \to \infty$.

Introduce a chiral Neveu-Schwarz fermion living in the rapidity complex plane and having mode expansion

$$\psi(u) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \psi_r \, u^{-r-\frac{1}{2}}, \quad \bar{\psi}(u) = \sum_{r\in\mathbb{Z}+\frac{1}{2}} \bar{\psi}_r \, u^{r-\frac{1}{2}}. \tag{3.1}$$

The fermion modes are assumed to satisfy the anticommutation relations

$$[\bar{\psi}_r, \bar{\psi}_{r'}]_+ = [\psi_r, \psi_{r'}]_+ = 0\,, \quad [\psi_r, \bar{\psi}_{r'}]_+ = \delta_{r,r'}\,, \tag{3.2}$$

and the left/right vacuum states are defined by

$$\langle 0|\psi_{-r} = \langle 0|\bar{\psi}_r = 0 \;\text{ and }\; \psi_r\,|0\rangle = \bar{\psi}_{-r}|0\rangle = 0, \quad \text{for } r > 0. \tag{3.3}$$

The operator $\bar{\psi}_r$ creates a particle (or annihilates a hole) with mode number r and the operator ψ_r annihilates a particle (or creates a hole) with mode number r. The particles carry charge 1, while the holes carry charge -1. The charge zero vacuum states (3.3) are obtained by filling the Dirac see up to level zero. The left vacuum states with integer charge $\pm N$ are constructed as

$$\langle N| \;=\; \begin{cases} \langle 0|\psi_{\frac{1}{2}} \ldots \psi_{N-\frac{1}{2}} & \text{if } N > 0, \\ \langle 0|\bar{\psi}_{-\frac{1}{2}} \ldots \bar{\psi}_{-N+\frac{1}{2}} & \text{if } N < 0. \end{cases} \tag{3.4}$$

Any correlation function of the operators (3.1) is a determinant of two-point correlators

$$\langle 0|\psi(u)\bar{\psi}(v)|0\rangle = \frac{1}{u-v}\,. \tag{3.5}$$

Obviously, the Slavnov kernel (2.27) can be represented as the correlation function of two fermionic operators, located at the points u and v of the rapidity plane,

$$\Omega(u, v) \;=\; \langle 0| \left[\bar{\psi}(v) - e^{2ip_{\mathbf{u}}(v)}\,\bar{\psi}(v+i)\right] \left[\psi(u) - \psi(u+i)\right] |0\rangle. \tag{3.6}$$

The determinant of the matrix $\Omega(u_j, v_k)$ is equal to the correlation function of N pairs of such operators, and the Slavnov inner product (2.28) can be given the following Fock space representation,

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} \;=\; \frac{\langle 0| \prod_{j=1}^N \left[\psi(u_j) - \psi(u_j+i)\right] \prod_{k=1}^N \left[\bar{\psi}(v_k) - e^{2ip_{\mathbf{u}}(v_k)}\,\bar{\psi}(v_k+i)\right] |0\rangle}{\langle 0| \prod_{j=1}^N \psi(u_j+i) \prod_{k=1}^N \bar{\psi}(v_k)|0\rangle}. \tag{3.7}$$

Our aim is to rewrite (3.7) in a form convenient for taking the limit $N \to \infty$. For that we first we transform the denominator, using the Cauchy identity, to

$$\begin{aligned} \mathcal{K}_{\mathbf{u},\mathbf{v}} &= \langle 0| \prod_{j=1}^{N} \psi(u_j + i) \prod_{k=1}^{N} \bar{\psi}(v_k)|0\rangle \\ &= \det_{jk} \frac{1}{u_j - v_k + i} \\ &= \frac{\Delta_{\mathbf{u}}\ \Delta_{-\mathbf{v}}}{\prod_{j,k=1}^{N}(u_j - v_k + i)}. \end{aligned} \tag{3.8}$$

Here and below we denote by $\Delta_{\mathbf{w}}$ the Vandermonde determinant associated with the set of complex numbers $\mathbf{w} = \{w_1, \ldots, w_N\}$,

$$\Delta_{\mathbf{w}} \stackrel{\text{def}}{=} \prod_{j<k}^{N}(w_j - w_k). \tag{3.9}$$

Then we represent the expectation value in the numerator of (3.7) as a product of difference operators acting on the Cauchy product (3.8). The result is

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} = \frac{1}{\mathcal{K}_{\mathbf{u},\mathbf{v}}} \prod_{j=1}^{N} \left(1 - e^{2ip_{\mathbf{u}}(v_j)}\, e^{i\partial/\partial v_j}\right) \left(e^{-i\partial/\partial u_j} - 1\right) \mathcal{K}_{\mathbf{u},\mathbf{v}}, \tag{3.10}$$

where $e^{i\partial/\partial u}$ denotes the shift operator $u \to u + i$. We can commute the denominator of the Cauchy product to the left, using the relations

$$\begin{aligned} e^{-i\partial/\partial u_j} \prod_{k,l=1}^{N} \frac{1}{u_k - v_l + i} &= E_{\mathbf{v}}^{+}(u_j) \prod_{k,l=1}^{N} \frac{1}{u_k - v_l + i}\, e^{-i\partial/\partial u_j}, \\ e^{i\partial/\partial v_j} \prod_{k,l=1}^{N} \frac{1}{u_k - v_l + i} &= E_{\mathbf{v}}^{-}(v_j) \prod_{k,l=1}^{N} \frac{1}{u_k - v_l + i}\, e^{i\partial/\partial v_j}, \end{aligned} \tag{3.11}$$

until it cancels its inverse. Here and below we denote

$$E_{\mathbf{u}}^{\pm}(v) = \frac{Q_{\mathbf{u}}(v \pm i)}{Q_{\mathbf{u}}(v)}, \quad E_{\mathbf{v}}^{\pm}(u) = \frac{Q_{\mathbf{v}}(u \pm i)}{Q_{\mathbf{v}}(u)}. \tag{3.12}$$

Now we can write (3.10) in a factorized operator form,

$$\begin{aligned} \mathscr{S}_{\mathbf{u},\mathbf{v}} = \ (-1)^N \ & \frac{1}{\Delta_{\mathbf{v}}} \prod_{j=1}^{N} \left(1 - e^{2ip_{\mathbf{u}}(v_j)} E_{\mathbf{u}}^{-}(v_j)\, e^{i\partial/\partial v_j}\right) \Delta_{\mathbf{v}} \\ & \times \frac{1}{\Delta_{\mathbf{u}}} \prod_{j=1}^{N} \left(1 - E_{\mathbf{v}}^{+}(u_j)\, e^{-i\partial/\partial u_j}\right) \Delta_{\mathbf{u}}\ \cdot 1. \end{aligned} \tag{3.13}$$

The factorization is not complete because we must commute the operators $e^{i\partial/\partial v_j}$ to the right through the functions $E_{\mathbf{v}}^{+}$ in the second factor, which depend implicitly of the v-variables:

$$e^{i\partial/\partial v_k}\ E_{\mathbf{v}}^{+}[u_j] = \left(1 - \frac{1}{(u_j - v_k)^2 + 1}\right) E_{\mathbf{v}}^{+}[u_j]\, e^{i\partial/\partial v_k}. \tag{3.14}$$

3.2 Another writing of the operator factorization

The operator representation (3.13) of the inner product of an on-shell state $|\mathbf{u}\,\rangle$ and an off-shell state $\langle\mathbf{v}|$ is the main tool we are going to use to investigate the classical limit of large L and N. Here we will give it a more abstract formulation in terms of a pair of non-commuting functional variables, which become c-functions in the classical limit.

For any set of points $\mathbf{u} = \{u_j\}_{j=1}^N$ in the complex plane and for any function $f(u)$ define the functional

$$\mathscr{A}^{\pm}_{\mathbf{u}}[f] \;\stackrel{\text{def}}{=}\; \frac{1}{\Delta_{\mathbf{u}}} \prod_{j=1}^{N} \left(1 - f(u_j)\, e^{\pm i\partial/\partial u_j}\right) \Delta_{\mathbf{u}}\,, \tag{3.15}$$

or, in terms of free fermions,

$$\begin{aligned} \mathscr{A}^{\pm}_{\mathbf{u}}[f] &= \frac{\langle N|\, \prod_{k=1}^N \left[\bar\psi(u_k) - f(u_k)\,\bar\psi(u_k \pm i)\right]|0\rangle}{\langle N|\prod_{k=1}^N \bar\psi(u_k)|0\rangle} \\ &= \frac{\det_{jk}\left(u_j^{k-1} - f(u_j)\,(u_j \pm i)^{k-1}\right)}{\det_{jk}\left(u_j^{k-1}\right)}\,. \end{aligned} \tag{3.16}$$

The functionals $\mathscr{A}^{\pm}_{\mathbf{u}}[f]$ are completely symmetric polynomials of the values of the function f on the set $\mathbf{u}$, having total degree N.

Assuming that $\mathbf{u}\cap\mathbf{v} = \emptyset$, we can write the r.h.s. of (3.13) as a matrix element

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} \;=\; (-1)^N\, \frac{(\mathbf{v}|\; \mathscr{A}^{+}_{\mathbf{v}}[U]\;\; \mathscr{A}^{-}_{\mathbf{u}}[V]\,|\mathbf{u})}{(\mathbf{v}|\mathbf{u})}\,, \tag{3.17}$$

where the operator functions $U(v)$ and $V(u)$ satisfy the algebra

$$U(v)V(u) = V(v)U(u)\, \left(1 - \frac{1}{(u-v)^2+1}\right) \tag{3.18}$$

and act on the vectors $(\mathbf{v}|$ and $|\mathbf{u})$ as

$$\begin{aligned} U(v)\,|\mathbf{u}) &= e^{2ip_{\mathbf{u}}(v)}\, E^{-}_{\mathbf{u}}(v)\,|\mathbf{u}) = \frac{d(v)}{a(v)} E^{+}_{\mathbf{u}}(v)\,|\mathbf{u})\,, \\ (\mathbf{v}|\,V(u) &= E^{+}_{\mathbf{v}}(u)\,(\mathbf{v}|. \end{aligned} \tag{3.19}$$

In this way the problem of evaluating the inner product reduces to the problem of evaluating the functionals (3.15).

3.3 Generalization to non-highest-weight states

The N-magnon states (2.12) are highest weight states with respect to the fully ordered state, pseudo-vacuum $|\!\uparrow^L\rangle$. Each operator $\mathbb{B}(u_j)$ flips one spin down so that the third component of the spin of the state with N magnons built on the pseudo vacuum $|0\rangle$ is $\mathbb{S}^z = \frac{1}{2}L - N$.

A complete system of states is obtained by acting with magnon creation operators on the vacuum descendant states $(\mathbb{S}^-)^K|\!\uparrow^L\rangle$ with $K \leq L/2$, which correspond to ferromagnetic vacua rotated away

from the third axis. On the other hand, the components $\mathbb{S}^{\pm}$ of the total spin can be obtained by taking the infinite rapidity limit of the magnon creation and annihilation operators,

$$\mathbb{B}(u) \simeq \frac{i}{u}\mathbb{S}^-, \quad \mathbb{C}(u) \simeq \frac{i}{u}\mathbb{S}^+. \tag{3.20}$$

The factor $1/u$ is obtained by comparing the large u asymptotics $\langle\uparrow^L|\,\mathbb{B}^\dagger(\bar{u})\mathbb{B}(u)\,|\uparrow^L\rangle \simeq L/u^2$, which follows from (2.34), with the normalization of the spin operator $\langle\uparrow^L|\mathbb{S}^+\mathbb{S}^-|\uparrow^L\rangle = L$. Therefore an M-magnon state built upon a vacuum descendent can be obtained as the limit of a N-magnon state (2.12) with $K = N - M$ of the rapidities sent to infinity:

$$\begin{aligned} \langle\uparrow^L|(\mathbb{S}^+)^{K'} \prod_{j=1}^{N-K'} \mathbb{C}(v_j)| &= \langle \mathbf{v}_{N-K'} \cup \infty^{K'}|, \\ \prod_{l=1}^{N-K} \mathbb{B}(u_k)\,(\mathbb{S}^-)^K|\uparrow^L\rangle &= |\mathbf{u}_{N-K} \cup \infty^K\rangle, \end{aligned} \tag{3.21}$$

where the limits should be taken sequentially according to the definition

$$|\mathbf{u}_{N-1} \cup \infty\rangle \stackrel{\text{def}}{=} \lim_{u_N\to\infty} \frac{u_N}{i}|\,\mathbf{u}_N\rangle = \mathbb{S}^-|\mathbf{u}_{N-1}\rangle\,. \tag{3.22}$$

To evaluate the inner products of such states, we need to compute the result of sending u_N to infinity in the functionals $\mathscr{A}^{\pm}_{\mathbf{u}}[f^{\pm}]$, assuming that the function $f^{\pm}(u)$ behaves at infinity as

$$f^{\pm}(u) \simeq e^{\mp iK_{\pm}/u}, \qquad u \to \infty. \tag{3.23}$$

From the definition (3.15) we find, taking into account that $\Delta_{\mathbf{u}_N} \simeq u_N^{N-1}\Delta_{\mathbf{u}_{N-1}}$,

$$\begin{aligned} \mathscr{A}^{\pm}_{\mathbf{u}_{N-1}\cup\infty}[f^{\pm}] &\stackrel{\text{def}}{=} \lim_{u_N\to\infty} \frac{u_N}{i}\mathscr{A}^{\pm}_{\mathbf{u}_N}[f^{\pm}] \\ &= \lim_{u_N\to\infty} \frac{1}{\Delta_{\mathbf{u}_{N-1}}} u_N^{-N+1}(1 - e^{\mp iK/u_N} e^{\pm i\partial/\partial u_N}) u_N^{N-1}\,\Delta_{\mathbf{u}_{N-1}} \\ &= \pm(K_{\pm} - N + 1)\,\mathcal{A}^{\pm}_{\mathbf{u}_{N-1}}[f^{\pm}]. \end{aligned} \tag{3.24}$$

Applying this relation the necessary number of times, one can obtain the generalization of the operator factorization formula (3.17) to the case of an inner product of non-highest-weight states.

3.4 The case when the Bethe eigenstate is a vacuum descendent

The inner product of Bethe state with a vacuum descendent is obtained from eq. (3.17) by sending all the roots from the set $\mathbf{u}$ to infinity. In this limit limit the sets $\mathbf{u}$ and $\mathbf{v}$ are infinitely separated, the commutator in (3.18) vanishes, and (3.19) gives $U(v) = \frac{d(v)}{a(v)}$ and $V(u) \sim e^{iN/u}$. Applying sequentially (3.24) to all variables $\mathbf{u}$, one obtains $\mathscr{A}^-_{\infty^N}[V] = (-1)^N N!$, and (3.17) gives

$$\langle\uparrow^L| \prod_{j=1}^N \mathbb{C}(v_j)\,(\mathbb{S}^-)^N|\uparrow^L\rangle = \mathscr{S}_{\infty^N,\mathbf{v}_N} = N!\,\mathscr{A}^+_{\mathbf{v}}[d/a]. \tag{3.25}$$

To evaluate the scalar product of two vacuum descendants, we have to send the set $\mathbf{v}$ to infinity as well. Applying sequentially (3.24), with $K_- = L$, to all variables from the set $\mathbf{v}$, one finds

$$\langle\uparrow^L|(\mathbb{S}^+)^N(\mathbb{S}^-)^N|\uparrow^L\rangle = \mathscr{S}_{\{\infty^N\},\{\infty^N\}} = (N!)^2\binom{L}{N}. \tag{3.26}$$

The second factor counts the number of ways to have N reversed spins in a chain of length L.

3.5 Two-kink pseudo-vacua and restricted scalar products

In view of the applications to SYM, we are going to consider pseudo-vacua with two kinks at distance K. In such a state the first K spins are down and the rest $L-K$ spins are oriented up:

$$\langle\downarrow^K\uparrow^{L-K}| = \langle\downarrow\downarrow \overset{K}{\cdots} \downarrow\uparrow\uparrow \overset{L-K}{\cdots} \uparrow|. \tag{3.27}$$

This state can be obtained by acting on the left pseudo-vacuum $\langle\uparrow^L|$ with K annihilation operators [30]

$$\langle\downarrow^K\uparrow^{L-K}| = \langle\uparrow^L| \prod_{j=1}^{K} \mathbb{C}(z_j) = \langle\!\langle \mathbf{z}_K |\!| \,, \tag{3.28}$$

with rapidities $\mathbf{z}_K = \{z_1, \dots, z_k\}$ determined by the values of the inhomogeneity parameters associated with the first K sites:

$$z_k \equiv \theta_k + \tfrac{1}{2}i \qquad (k = 1, \dots, K). \tag{3.29}$$

Hence, the inner products with the left pseudo-vacuum replaced by the two-kink state (3.27) are evaluated by restricting K of the rapidities[5]

$$\langle\downarrow^K\uparrow^{L-K}| \prod_{j=1}^{N-K} \mathbb{C}(v_j) \prod_{k=1}^{N} \mathbb{B}(u_k)|\uparrow^L\rangle = \langle\!\langle \mathbf{v}_{N-K} \cup \mathbf{z}_K \,|\, \mathbf{u}_N \rangle\!\rangle_{\boldsymbol{\theta}_L}, \tag{3.30}$$

with

$$\boldsymbol{\theta}_L = \boldsymbol{\theta}_{L-K} \cup \boldsymbol{\theta}_K \,, \qquad \mathbf{z}_K = \boldsymbol{\theta}_K - \tfrac{1}{2}i. \tag{3.31}$$

The restricted scalar product (3.30) can be computed by a factorized expression similar to (3.17),

$$\langle\!\langle \mathbf{v} \cup \mathbf{z} \,|\, \mathbf{u} \rangle\!\rangle_{\boldsymbol{\theta}} \;=\; (-1)^N \, \frac{(\mathbf{v}|\; \mathscr{A}^+_{\mathbf{v}}[U] \;\; \mathscr{A}^-_{\mathbf{u}}[V] \,|\mathbf{u})}{(\mathbf{v}|\mathbf{u})} \,, \tag{3.32}$$

where the operator functions $U(v)$ and $V(u)$ satisfy the algebra (3.18) and act on the vectors $(\mathbf{v}|$ and $|\mathbf{u})$ as

$$\begin{aligned} U(v)\,|\mathbf{u}) \;&= \kappa \, \frac{1}{E^+_{\mathbf{z}}(v)} \frac{Q_{\boldsymbol{\theta}}(v - \frac{1}{2}i)}{Q_{\boldsymbol{\theta}}(v + \frac{1}{2}i)} \; E^+_{\mathbf{u}}(v)|\mathbf{u}) \,, \\ (\mathbf{v}|\, V(u) \;&= E^+_{\mathbf{z}}(u) \; E^+_{\mathbf{v}}(u) \;\; (\mathbf{v}| \,. \end{aligned} \tag{3.33}$$

This trivial substitution shows the power of the operator expression (3.13). As a comparison, when evaluating the restricted scalar product using the original Slavnov determinant, one comes upon spurious singularities (poles and zeroes on the top of each other), which require multiple use of l'Hôpital's rule.

In the limit $\mathbf{u}, \mathbf{v} \to \infty$, defined as in (3.24), one obtains

$$\mathscr{S}_{\infty^N,\,\infty^{N-K}\cup\,\mathbf{z}_K} = N!(N-K)! \binom{L-K}{N}. \tag{3.34}$$

The second factor in (3.34) counts the number of non-equivalent ways to reverse N spins among the remaining $L-K$ up-spins of the partially ordered pseudo-vacuum.

[5]The restricted inner product (3.30) has been studied in [30,31,48,49]. A statistical interpretation of the restricted scalar product as partition functions of the six-vertex model is given in [31,49].

3.6 Gaudin-Izergin determinant and pDWPF

In the particular case $K = L = N$, the restricted scalar product

$$\mathscr{Z}_{\mathbf{u},\mathbf{z}} \equiv \langle \mathbf{z}_N | \mathbf{u}_N \rangle_{\boldsymbol{\theta}_N} = \langle \downarrow^N | \prod_{j=1}^{N} \mathbb{B}(u_j) \ | \uparrow^N \rangle , \qquad \mathbf{z}_N = \boldsymbol{\theta}_N + i/2 , \tag{3.35}$$

evaluates the partition function of the six-vertex model with domain-wall boundary conditions (DWBC) on a $N \times N$ square grid [27,50]. With this specialization of the rapidities, the second term of the kernel $\Omega(u, v)$, eq. (2.27), vanishes at $v = z_k$ and the Slavnov formula (2.28) gives

$$\mathscr{Z}_{\mathbf{u},\mathbf{z}} = \frac{\det_{jk} t(u_j - z_k)}{\det_{jk} \frac{1}{u_j - z_k + i}} , \quad t(u) = \frac{1}{u} - \frac{1}{u+i} . \tag{3.36}$$

The determinant representation (3.36) of the DWBC partition function was obtained by Izergin [51, 52]. For the first time the ratio of determinants (3.36) appeared in the works of M. Gaudin [38,53] as the scalar product of two Bethe wave functions for a Bose gas with point-like interaction on an infinite line.

The Gaudin-Izergin determinant (3.36) has an obvious writing in terms of free fermions,

$$\mathscr{Z}_{\mathbf{u},\mathbf{z}} = \frac{\langle 0 | \prod_{j=1}^{N} [\psi(u_j) - \psi(u_j + i)] \ \prod_{k=1}^{N} \bar{\psi}(z_k) | 0 \rangle}{\langle 0 | \prod_{j=1}^{N} \psi(u_j + i) \prod_{k=1}^{N} \bar{\psi}(z_k) | 0 \rangle}, \tag{3.37}$$

and is expressed in terms of the functionals (3.15) as

$$\mathscr{Z}_{\mathbf{u},\mathbf{z}} = (-1)^N \mathscr{A}^-_{\mathbf{u}}[E^+_{\mathbf{z}}]. \tag{3.38}$$

The limit of $\mathscr{Z}_{\mathbf{u},\mathbf{z}}$ when part of the rapidities $\mathbf{u}$ are sent to infinity was recently studied by Foda and Wheeler in [47] and given the name partial domain-wall partition function, or pDWPF. We compute the pDWPF by applying sequentially (3.24) to $N - n$ of the variables, with the result

$$\mathscr{Z}_{\mathbf{u}_n \cup \infty^{N-n}, \mathbf{z}_N} = (-1)^n (N-n)! \ \mathscr{A}^-_{\mathbf{u}_n}[E^+_{\mathbf{z}_N}] . \tag{3.39}$$

An alternative proof of eq. (3.39) is presented in [47]. The fermionic representation of pDWPF is

$$\mathscr{Z}_{\mathbf{u}_n \cup \infty^{N-n}, \mathbf{z}_N} = (N-n)! \ \frac{\langle N-n | \prod_{j=1}^{n} [\psi(u_j) - \psi(u_j + i)] \ \prod_{k=1}^{N} \bar{\psi}(z_k) | 0 \rangle}{\langle N-n | \prod_{j=1}^{n} \psi(u_j + i) \prod_{k=1}^{N} \bar{\psi}(z_k) | 0 \rangle}. \tag{3.40}$$

The expectation value on the r.h.s. of (3.40) is defined for any pair of non-negative integers N and n, but it vanishes identically when $n > N$, This yields a pair of identities

$$\mathscr{A}^{\pm}_{\mathbf{u}_n}[E^{\mp}_{\mathbf{z}_N}] = 0 \quad \text{for } N < n. \tag{3.41}$$

3.7 Properties of the functionals $\mathscr{A}^{\pm}_{\mathbf{u}}[f]$

Expansions

The functionals $\mathscr{A}^{\pm}_{\mathbf{u}}$, defined by eq. (3.15), are obviously completely symmetric polynomials of degree N of the variables $f(u_1), \ldots, f(u_N)$. The coefficients of the polynomial are obtained by

expanding the product in (3.15) as a sum of monomials labeled by all possible partitions of the set $\mathbf{u}$ into two disjoint subsets $\mathbf{u}'$ and $\mathbf{u}''$, with $\mathbf{u}' \cup \mathbf{u}'' = \mathbf{u}$,

$$\mathscr{A}_{\mathbf{u}}^{\pm} = \sum_{\mathbf{u}'\cup\mathbf{u}''=\mathbf{u}} (-1)^{|\mathbf{u}'|} \left(\prod_{u'\in\mathbf{u}'} f(u')\right) \frac{1}{\Delta_{\mathbf{u}}} \prod_{u'\in\mathbf{u}'} e^{\pm i\partial/\partial u'} \Delta_{\mathbf{u}}. \tag{3.42}$$

Here $|\mathbf{u}'|$ stands for the number of elements of the subset $\mathbf{u}'$. The last factor is evaluated as

$$\frac{1}{\Delta_{\mathbf{u}}} \left(\prod_{u'\in\mathbf{u}'} e^{\pm i\partial/\partial u'}\right) \Delta_{\mathbf{u}} = \prod_{u'\in\mathbf{u}', u''\in\mathbf{u}''} \frac{u'-u''\pm i}{u'-u''}. \tag{3.43}$$

The expansion (3.42) can be used as an alternative definition of the functional $\mathscr{A}_{\mathbf{u}}^{\pm}[f]$. For $f = d/a$, this expansion was thoroughly studied by Gromov, Sever and Vieira [11]. It was found in [11] that for constant function $f(u) = \kappa$, the expansion (3.42) does not depend on the positions of the rapidities $\mathbf{u}$ and the functional $\mathscr{A}_{\mathbf{u}}^{\pm}[f]$ is given in this case by

$$\mathscr{A}_{\mathbf{u}}^{\pm}[\kappa] = (1-\kappa)^N = \exp\left(-N\sum_{n=1}^{\infty}\frac{\kappa^n}{n}\right). \tag{3.44}$$

2. The linear term in f as a contour integral

The linear term in f can be evaluated as a contour integral:

$$\begin{aligned} \mathscr{A}_{\mathbf{u}}^{\pm}[f] &= 1 - \sum_{j=1}^{N} f(u_j) \prod_{k(\neq j)} \frac{u_j-u_k\pm i}{u_j-u_k} + O[f^2] \\ &= 1 \pm \oint_{A_{\mathbf{u}}} \frac{du}{2\pi}\, f(u) E_{\mathbf{u}}^{\pm}(u) + O[f^2]. \end{aligned} \tag{3.45}$$

The integration contour $A_{\mathbf{u}}$ encircles all points of the set $\mathbf{u}$ and leaves outside the possible singularities of the function $f(u)$.

Functional identities

Using the fermionic representation (3.16) and the fact that the fermion correlator is translation invariant, we transform

$$\begin{aligned} \mathscr{A}_{\mathbf{u}}^{+}[f] &= \frac{\langle N|\ \prod_{k=1}^{N}\left[\bar\psi(v_k-i) - f(v_k)\,\bar\psi(v_k)\right]|0\rangle}{\langle N|\prod_{k=1}^{N}\bar\psi(v_k)|0\rangle}. \\ &= (-1)^N f(v_1)\dots f(v_N)\, \frac{\langle 0|\prod_{k=1}^{N}\left[\bar\psi(v_k) - \frac{1}{f(v_k)}\,\bar\psi(v_k-i)\right]|N\rangle}{\langle 0|\prod_{k=1}^{N}\bar\psi(v_k)|N\rangle} \\ &= (-1)^N f(v_1)\dots f(v_N)\, \mathscr{A}_{\mathbf{u}}^{-}[1/f]. \end{aligned} \tag{3.46}$$

Hence, $\mathscr{A}^+$ and $\mathscr{A}^-$ are related by the functional identities

$$\mathscr{A}_{\mathbf{u}}^{\pm}[1/f] = (-1)^N \frac{\mathscr{A}_{\mathbf{u}}^{\mp}[f]}{\prod_{j=1}^{N} f(u_j)}. \tag{3.47}$$

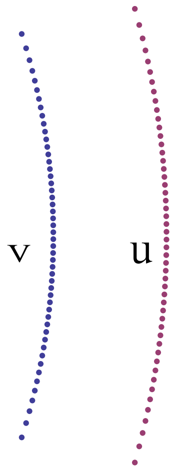

Figure 1: An example of the distributions $\mathbf{u}$ and $\mathbf{v}$ for $N = 50$, each consisting of a single macroscopic Bethe string.

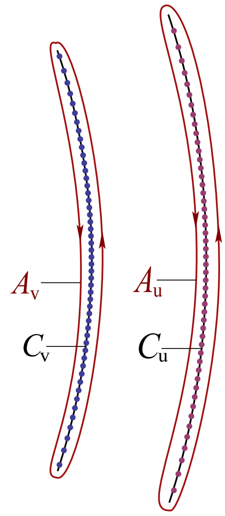

Figure 2: The cuts $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ and the integration contours $A_{\mathbf{u}}$ and $A_{\mathbf{v}}$ for the one-cut solution of Fig. 1.

4 Classical limit

In this section we will find the classical limit of the inner product (2.28). The classical limit is achieved when $L, N \to \infty$ with $\alpha = N/L$ finite, and some additional assumptions on the distribution of the rapidities $\mathbf{u}_N$ and $\mathbf{v}_N$. In the condensed matter literature the classical limit, in which each Bethe string has macroscopic number of particles, has been studied by Sutherland [34] and by Dhar and Shastry [54]. In this regime the roots $\mathbf{u}$ condense along a curve $C_{\mathbf{u}}$ in the rapidity plane, consisting of several connected components $C_{\mathbf{u}_1}, \dots, C_{\mathbf{u}_n}$, with $\mathbf{u}_1 \cup \dots \mathbf{u}_n = \mathbf{u}$, symmetric about the real axis, with slowly varying linear density $\rho_{\mathbf{u}}(u)$ [12]. The curve $C^k_{\mathbf{u}}$ contains $N_k = \#\mathbf{u}_k$ particles,

$$\int_{C^k_{\mathbf{u}}} \rho(u) du = N_k \,, \qquad N_1 + \cdots + N_n = N. \tag{4.1}$$

We assume that the filling fractions $\alpha_k = N_k/L$ associated with the cuts C_k remain finite when $L \to \infty$. Then the size of each curve is $\sim L$. We assume a similar behavior for the rapidities $\mathbf{v}$. An example of distributions $\mathbf{u}$ and $\mathbf{v}$ with $N = 50$ and $n = 1$ is given in Fig. 1.

In the classical limit, the arguments U and V in Eq. (3.17) become c-functions, and the inner product factorizes to

$$\mathscr{S}_{\mathbf{u},\mathbf{v}} \;=\; (-1)^N \, \mathscr{A}^+_{\mathbf{v}}[\kappa \, e^{iG_{\mathbf{u}} - iG_{\boldsymbol{\theta}}}] \; \mathscr{A}^-_{\mathbf{u}}[e^{iG_{\mathbf{v}}}] \,, \tag{4.2}$$

where

$$G_{\mathbf{u}}(u) = \partial_u \log Q_{\mathbf{u}}(u), \quad G_{\mathbf{v}}(u) = \partial_u \log Q_{\mathbf{v}}(u), \quad G_{\boldsymbol{\theta}}(u) = \partial_u \log Q_{\boldsymbol{\theta}}(u) \tag{4.3}$$

are the resolvents associated respectively with the sets $\mathbf{u}$, $\mathbf{v}$ and $\boldsymbol{\theta}$. The resolvent $G_{\mathbf{u}}(u)$ is a meromorphic function of u with cuts $C^1_{\mathbf{u}}, \dots, C^n_{\mathbf{u}}$ and asymptotics N/u at infinity. The discontinuity across the cuts is proportional to the density $\rho_{\mathbf{u}}(u)$.

The form of the curve $C_{\mathbf{u}}$ and the density of the distribution of the roots is determined by the finite-zone solution constructed in [12], which we discrete shortly below. In the vicinity of each cut $C_{\mathbf{u}}^k$, the pseudo-momentum

$$p(u) = G_{\mathbf{u}}(u) - \tfrac{1}{2}G_{\boldsymbol{\theta}}(u) + \tfrac{1}{2}\phi \,(\mathrm{mod}\, \pi) \tag{4.4}$$

splits into a continuous part $\not{p}(u)$, equal to the half-sum of the values of $p(u)$ on both sides of $C_{\mathbf{u}}$ and a discontinuous part $\hat{p}(u)$, which is proportional to the density $\rho_{\mathbf{u}}$:

$$p_{\mathbf{u}}(u) = \not{p}_{\mathbf{u}}(u) + \hat{p}_{\mathbf{u}}(u), \quad |\hat{p}_{\mathbf{u}}(u)| = \pi\rho_{\mathbf{u}}(u). \tag{4.5}$$

When the set $\mathbf{u}$ satisfies the Bethe equations, the eigenvalue of the transfer matrix $T_{\mathbf{u}} = 2\cos p_{\mathbf{u}}$ is analytic in u and hence takes the same value on both sides of the cuts $C_{\mathbf{u}}^k$. This yields the boundary condition $\sin \hat{p}_{\mathbf{u}}\, \sin \not{p} = 0$ on the cuts. Along each cut $\rho_{\mathbf{u}} > 0$, which implies $\not{p}_{\mathbf{u}} = 0 \;(\mathrm{mod}\, \pi)$, or

$$\not{p}(u) = \pi n_k, \qquad u \in C_{\mathbf{u}}^k, \tag{4.6}$$

where n_k is the mode number associated with the cut $C_{\mathbf{u}}^k$. The branch points are zeroes of the entire function $\Delta(u) \equiv T_{\mathbf{u}}^2(u) - 4 = -4\sin^2 p_{\mathbf{u}}$. The forbidden zones $\Delta(u) > 0$ are associated with the cuts of the pseudo-momentum $p(u)$ on the first sheet.

The typical situation in the homogeneous limit is when $\Delta(u)$ has a double zero at $u = \infty$, $2n$ simple zeroes $a_1, \bar{a}_1, \ldots, a_n, \bar{a}_n$, and infinitely many negative double zeros at $a_{-1}, a_{-2}, \ldots a_{-k}, \ldots$ where $p(a_{-k}) = 2\pi n_{-k}$, accumulating at the point of essential singularity $u = 0$. The cuts are along the forbidden zones between a_k and $\bar{a}_k$ $(k = 1, \ldots, n)$.

The derivative of pseudomomentum $\partial p_{\mathbf{u}}(u)$ as well as the exponential $e^{2ip(u)}$ are defined on a Riemann surface associated with the hyper-elliptic complex curve

$$y^2 = \prod_{k=1}^{n} (u^2 - a_k^2)\,. \tag{4.7}$$

The values of $e^{ip(u)}$ on the first and on the second sheet are related by

$$e^{ip^{(1)}(u)} = e^{-ip^{(2)}(u)}. \tag{4.8}$$

The derivatives of the pseudo-momentum in N_j,

$$\partial_{N_j} p(u) du = \omega_j(u), \quad j = 1, \ldots, n \tag{4.9}$$

form a basis of holomorphic abelian differentials on the complex curve curve:

$$\frac{1}{2\pi i}\oint_{A_{\mathbf{u}}^k} \omega_j = \delta_{kj}, \qquad k, j = 1, ..., n. \tag{4.10}$$

The cycle $A_{\mathbf{u}}^k$ represents a closed contour on the Riemann surface which encircles the cut $C_{\mathbf{u}}^k$ anti-clockwise, as shown in Fig. 2.

By the factorization formula (4.2), the computation of the inner product in the quasiclassical limit reduces to that of the functionals $\mathscr{A}_{\mathbf{u}}^{\pm}$. This last problem was solved in the particular case $f = d/a$ in [11]. Below we give an alternative, and a little bit shorter, derivation of their result. Let us remark that although both derivations seem rather convincing, neither of them is a rigorous one.

4.1 Classical limit of the functionals $\mathscr{A}^{\pm}_{\mathbf{u}}[f]$

Assume that the rapidities $\mathbf{u}$ are distributed along a (possibly milti-component) contour $C_{\mathbf{u}}$ in the complex u-plane with finite and sufficiently smooth density. When the functional argument f is small, by eq. (3.45),

$$\log \mathscr{A}^{\pm}_{\mathbf{u}}[f] \;=\; \pm \oint\limits_{A_{\mathbf{u}}} \frac{dz}{2\pi}\, e^{iq^{\pm}(z)} \;+\; O[f^2], \tag{4.11}$$

where the integration contour $A_{\mathbf{u}}$ encircles $C_{\mathbf{u}}$ anticlockwise, and the function $q^{\pm}(u)$ is defined by

$$q^{\pm}(u) = -i\log[f(u)] \pm G_{\mathbf{u}}(u). \tag{4.12}$$

On the other hand, the functional relations (3.47) allow us to determine the asymptotics for large f, which can be written again in the form of a contour integral. To see that we first express

$$\begin{aligned} \log\left[(-1)^N \prod_{j=1}^{N} f(u_j)\right] \;&=\; \oint\limits_{A_{\mathbf{u}}} \frac{du}{2\pi i}\, G_{\mathbf{u}}(u)\,(\log[f(u)]+i\pi) \\ &=\; \pm \oint\limits_{A_{\mathbf{u}}} \frac{du}{2\pi}\, [\tfrac{1}{2} q_{\pm}^2(u) + i\pi q_{\pm}(u)]. \end{aligned} \tag{4.13}$$

Substituting (4.13) in (3.47), we find the large f asymptotics

$$\log \mathscr{A}^{\pm}_{\mathbf{u}}[f] \simeq \oint\limits_{A_{\mathbf{u}}} \frac{du}{2\pi}\big(\pm\tfrac{1}{2}[q_{\pm}(u)+i\pi]^2 \mp e^{-iq^{\pm}(u)}\big) \;+\; O(f^{-2}]. \tag{4.14}$$

We will look for a solution compatible with the behavior at $f\to 0,\infty$, which should be of the form

$$\log \mathscr{A}^{\pm}_{\mathbf{u}}[f] = \oint_{A_{\mathbf{u}}} \frac{du}{2\pi}\, F^{\pm}\big(e^{iq^{\pm}(u)}\big), \tag{4.15}$$

where $q(z)$ is defined by (4.12) and the meromorphic function $F(\omega)$ has asymptotics

$$F^{\pm}(\omega) \simeq \begin{cases} \pm\omega + O(\omega^2) & \text{if } |\omega| \ll 1, \\ \mp\frac{1}{2}\log(-\omega)^2 \mp 1/\omega + O(1/\omega^2) & \text{if } |\omega| \gg 1. \end{cases} \tag{4.16}$$

The function $F(\omega)$ can be determined completely by comparing the Ansatz (4.15) with the known exact solution (3.44) for f constant. Assume that the function $F^{\pm}(\omega)$ is expanded in a Taylor series

$$F^{\pm}(\omega) = \sum_{n\geq 1} F^{\pm}_n\, \omega^n \tag{4.17}$$

in some vicinity of the point $\omega = 0$ and compute the r.h.s. of (4.15) for $q^{\pm}(u) = -i\log\kappa \pm G_{\mathbf{u}}(u)$, which corresponds to $f(u) = \kappa$. The contour integral can be evaluated by expanding the contour to infinity, and we find

$$\sum_n F^{\pm}_n \oint \frac{du}{2\pi}\, e^{\pm inq(u)} = \sum_n F^{\pm}_n \oint \frac{du}{2\pi}\, \kappa^n\, (1 \pm i\frac{N}{u} + \dots)^n = \mp \sum_n F^{\pm}_n\, nN\kappa^n. \tag{4.18}$$

Comparing (4.18) and (3.44) we find that $F_n^{\pm} = \pm 1/n^2$ and that the Taylor expansion (4.17) is that of the dilogarithm,

$$F^{\pm}(\omega) = \pm \sum_{n=1}^{\infty} \frac{\omega^n}{n^2} = \pm\,\mathrm{Li}_2(\omega). \tag{4.19}$$

The asymptotic behavior (4.16) is satisfied thanks to the functional equation for the dilogarithm,

$$\mathrm{Li}_2(\frac{1}{\omega}) = -\mathrm{Li}_2(\omega) - \frac{\pi^2}{6} - \frac{1}{2}\log^2(-\omega). \tag{4.20}$$

Moreover, the property (4.20) of the dilogarithm leads to a pair of functional equations for $\mathscr{A}_{\mathbf{u}}^{\pm}[f]$, which are the scaling limit of (3.47).

If the resolvent $G_{\mathbf{u}}$ has several cuts on $C_{\mathbf{u}}^1, \ldots, C_{\mathbf{u}}^n$, then the integration contour in (4.15) splits into n disjoint contours $A_{\mathbf{u}}^1, \ldots, A_{\mathbf{u}}^n$, and the functional $\mathscr{A}_{\mathbf{u}}^{\pm}[f]$ is given in the classical limit by

$$\log \mathscr{A}_{\mathbf{u}}^{\pm}[f] \;\; \simeq \;\; \pm \oint_{A_{\mathbf{u}}} \frac{du}{2\pi}\, \mathrm{Li}_2\big(f(u)\, e^{\pm i G_{\mathbf{u}}(u)}\big)\,, \quad A_{\mathbf{u}} = \cup_{k=1}^{n} A_{\mathbf{u}}^k. \tag{4.21}$$

The k-th term grows as $\alpha_k L$, where $\alpha_k = N_k/L$ is the filling fraction associated with the cut $C_{\mathbf{u}}^k$. Let us emphasize that in the derivation of (4.21) we did not assume that the set $\mathbf{u}$ is on-shell.

Finally, let us note that the functional identity (3.47), or equivalently, the property of the dilogarithm (4.20), leads to a second integral representation,

$$\log \mathscr{A}_{\mathbf{u}}^{\pm}[f] \simeq \mp \oint_{A_{\mathbf{u}}} \frac{du}{2\pi}\, \mathrm{Li}_2\big(f^{-1}(u)\, e^{\mp i G_{\mathbf{u}}(u)}\big) + \oint_{A_{\mathbf{u}}} \frac{du}{2\pi} G_{\mathbf{u}}(u) \log f(u) \, + i\pi N. \tag{4.22}$$

4.2 Classical limit of the Slavnov inner product $\mathscr{S}_{\mathbf{u},\mathbf{v}}$

Substituting (4.22) in (4.2), we write the logarithm of the scalar product as a contour integral

$$\log \mathscr{S}_{\mathbf{u},\mathbf{v}} = i\pi N + \oint_{A_{\mathbf{v}}} \frac{du}{2\pi}\, \mathrm{Li}_2\left(\kappa\, e^{iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u) - iG_{\boldsymbol{\theta}}(u)}\right) - \oint_{A_{\mathbf{u}}} \frac{du}{2\pi}\, \mathrm{Li}_2\left(e^{-iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u)}\right). \tag{4.23}$$

The r.h.s. of (4.23) can be reformulated in terms of a contour integral around the ensemble of the cuts of the function

$$q(u) \;\; \stackrel{\text{def}}{=} \;\; G_{\mathbf{u}}(u) + G_{\mathbf{v}}(u) - G_{\theta}(u) + \phi \tag{4.24}$$

on the physical sheet of its Riemann surface, with integrand depending only on $q(u)$. This follows from the fact that the resolvent $G_{\mathbf{u}}$ satisfies on its cuts $C_{\mathbf{u}}^k$ the boundary condition (4.6),

$$2\not{G}_{\mathbf{u}}(u) - G_{\boldsymbol{\theta}}(u) + \phi = 2\pi n_k \;\text{ for }\; u \in C_{\mathbf{u}}^k. \tag{4.25}$$

(Here $\not{G}_{\mathbf{u}}$ is the half-sum of the values of the resolvent on both sides of $C_{\mathbf{u}}^k$ and $n_k \in \mathbb{Z}$ is the corresponding mode number.) Hence, if $q^{(1)}$ is the value of the function $q(z)$ on the physical sheet, defined by (4.24), then the value of $q(u)$ on the second sheet under the cut $C_{\mathbf{u}}^k$ is given by $q^{(2)} = -G_{\mathbf{u}} + G_{\mathbf{v}}$. We conclude that the two integrals in (4.23) have the same integrand $\mathrm{Li}_2(e^{iq})$, but the

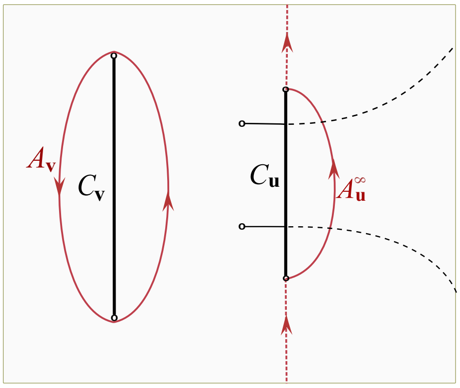

Figure 3: The contour $A_{\mathbf{v}}$ and the deformed contour $A_{\mathbf{u}}^{\infty}$ for the integral in (4.26). The contour $A_{\mathbf{u}}^{\infty}$ starts at $z = -i\infty$ on the second sheet, passes on the first sheet at the lower branch point, continues to the upper branch point, where it returns to the second sheet and continues to $z = +i\infty$.

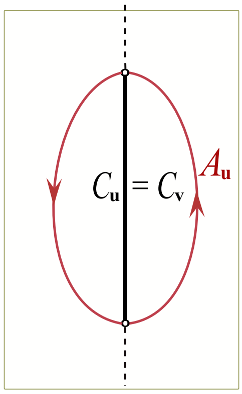

Figure 4: The contour $A_{\mathbf{u}}$ for the integral in (4.27), obtained as the limit $\mathbf{v} \to \mathbf{u}$ of the integral (4.26). When $C_{\mathbf{v}} \to C_{\mathbf{u}}$, the two logarithmic branch points on the first sheet join the two simple branch points at the ends of the cut $C_{\mathbf{u}}$.

contours $A_{\mathbf{u}}$ are placed on the second sheet of the Riemann surface of the function $q(u)$. After pulling all connected components $A_{\mathbf{u}}^k \subset A_{\mathbf{u}}$ up to the first sheet across the cuts $C_{\mathbf{u}}^k$, eq. (4.23) takes the form

$$\log \mathscr{S}_{\mathbf{u},\mathbf{v}} \;=\; i\pi N + \oint\limits_{A_{\mathbf{u}}\cup A_{\mathbf{v}}} \frac{du}{2\pi}\, \mathrm{Li}_2(e^{i\,q(u)}). \tag{4.26}$$

(The minus sign is compensated by the change of the orientation of the contours $A_{\mathbf{u}}^k$ after they are moved to the first sheet.)

The integral along $A_{\mathbf{u}}^k$ is however ambiguous, because the integrand has two logarithmic cuts which start at $u = \infty$ on the second sheet, cross the cut $C_{\mathbf{u}}^k$ and end at two branch points on the first sheet. The ambiguity is resolved by deforming the contour $A_{\mathbf{u}}$ to a contour $A_{\mathbf{u}}^{\infty}$ which encircles also the point $z = \infty$ on the second sheet.[6] In the case of a one-cut solution, the contour $A_{\mathbf{u}}^{\infty}$ is depicted in Fig. 3.

In the limit $\mathbf{u} \to \infty$, eq. (4.26) must reproduce the result of [11] about the scalar product of two general Bethe states and a vacuum descendent (eqs. (3.28)-(3.29) of [11]). This is indeed the case. In the limit $\mathbf{u} \to \infty$, the integration goes only along the contour $C_{\mathbf{v}}$ and the function q in the integrand is given in the homogeneous limit by $q = G_{\mathbf{v}} - \frac{L}{u} + \phi$.

4.3 Classical limit of the Gaudin norm

An expression for the square of the Gaudin norm in the classical limit can be formally obtained from (4.26) by taking $G_{\mathbf{v}} = G_{\mathbf{u}}$. Here we assumed that the two sets of rapidities are invariant under complex conjugation, $\bar{\mathbf{u}} = \mathbf{u}$ and $\bar{\mathbf{v}} = \mathbf{v}$. When $\mathbf{v} \to \mathbf{u}$, the integration contour $A_{\mathbf{u}}$ in (4.26) can be

[6] The author is indebted to Nikolay Gromov for performing a numerical test and for suggesting how to place the integration contours.

closed around $C_{\mathbf{u}}$ and $C_{\mathbf{v}}$ as in Fig. 4, and in the integrand one can replace $q(z) \to 2p(z)$, where $p(u)$ is the quasi-momentum (4.4). We find for the square of the Gaudin norm $\langle \mathbf{u}|\mathbf{u}\rangle^2 = \mathscr{S}_{\mathbf{u},\mathbf{u}}$, with

$$\log \mathscr{S}_{\mathbf{u},\mathbf{u}} \;\; = \;\; \oint\limits_{A_{\mathbf{u}}} \frac{du}{2\pi}\, \mathrm{Li}_2 \left(e^{2ip(u)} \right). \tag{4.27}$$

The term $i\pi N$ is compensated by another such term which appears because of the hermitian conjugation $\mathcal{B}^\dagger = -\mathcal{C}$. Furthermore, when $C^k_{\mathbf{v}} = C^k_{\mathbf{u}}$, the two cuts on the second sheet end at the two branch points where $1 - e^{2ip_{\mathbf{u}}} = 2ie^{ip_{\mathbf{u}}} \sin p_{\mathbf{u}} = 0$ and $\mathrm{Li}_2(1 - e^{2ip_{\mathbf{u}}})$ has a logarithmic singularity $\mathrm{Li}_2 \sim \sin(p_{\mathbf{u}}) \log \sin(p_{\mathbf{u}})$. Therefore there is no obstructions for placing the integration contour $A_{\mathbf{u}}$.

An expression of the Gaudin norm as a linear integral was derived in [11]. One can check, using the fact that $p_{\mathbf{u}}(z) = \pm i\pi\rho_{\mathbf{u}}(z)$ on the two edges of the cut, that the contour integral (4.27) can be transformed into (twice) the linear integral in eq. (2.15) of [11].

4.4 Classical limit of the restricted inner product

From Eqs. (3.32)-(3.33), one readily evaluates the classical limit of the restricted scalar product (3.30),

$$\mathscr{S}_{\mathbf{u},\mathbf{v}\cup\mathbf{z}} \;\; = \;\; (-1)^N \; \mathscr{A}^+_{\mathbf{v}\cup\mathbf{z}}[\kappa\, e^{iG_{\mathbf{u}} - iG_{\boldsymbol{\theta}} - iG_{\mathbf{z}}}] \;\; \mathscr{A}^-_{\mathbf{u}}[e^{iG_{\mathbf{v}} + iG_{\mathbf{z}}}], \tag{4.28}$$

where $\boldsymbol{\theta} \cup \mathbf{z}$ are the inhomogeneity parameters, and then use (4.21). We find

$$\begin{aligned} \log \mathscr{S}_{\mathbf{u},\mathbf{v}\cup\mathbf{z}} \;\; &= \;\; i\pi N + \oint\limits_{A_{\mathbf{v}}} \frac{du}{2\pi}\, \mathrm{Li}_2 \left(\kappa\, e^{iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u) - iG_{\boldsymbol{\theta}}(u)} \right) - \oint\limits_{A_{\mathbf{u}}} \frac{du}{2\pi}\, \mathrm{Li}_2 \left(e^{-iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u) + iG_{\mathbf{z}}(u)} \right) \\ &= \;\; i\pi N + \oint\limits_{A_{\mathbf{u}} \cup A_{\mathbf{v}}} \frac{du}{2\pi}\, \mathrm{Li}_2 \left(e^{iG_{\mathbf{u}}(u) + iG_{\mathbf{v}}(u) - iG_{\boldsymbol{\theta}}(u)} \right). \end{aligned} \tag{4.29}$$

In the second line we used the classical Bethe equation for $\phi = 0$

$$2\not{G}_{\mathbf{u}}(u) - G_{\boldsymbol{\theta}}(u) - G_{\mathbf{z}}(u) = 0 \, (\mathrm{mod}\, 2\pi), \quad u \in C_{\mathbf{u}}. \tag{4.30}$$

We see that the restricted scalar product $\mathscr{S}_{\mathbf{u},\mathbf{v}\cup\mathbf{z}}$ is given by the same contour integral (4.26), where the integrand depends only on $\mathbf{u}$, $\mathbf{v}$ and $\boldsymbol{\theta}$. The dependence on $\mathbf{z}$ is through the boundary condition on the cuts of the resolvent $G_{\mathbf{u}}$.

4.5 The derivatives in N_k

Knowing the hyper-elliptic Riemann surface (4.7), it is possible to compute the logarithmic derivatives of the scalar product with respect to the filling numbers N_k. For that we use the fact that, according to (4.9), the derivatives of the pseudo-momentum form a basis of abelian differentials for the Riemann surface.

Take for example the Gaudin norm. From (4.27) we find for its logarithmic derivative

$$\begin{aligned} \frac{\partial \log \mathscr{S}_{\mathbf{u},\mathbf{u}}}{\partial N_k} &= -2i \oint\limits_{C_{\mathbf{u}}} \frac{dz}{2\pi} \frac{\partial p_{\mathbf{u}}}{\partial N_k} \; \log \left(1 - e^{2ip_{\mathbf{u}}(z)} \right) \\ &= -\frac{i}{\pi} \oint\limits_{C_{\mathbf{u}}} \omega_k(z) \; \log \left(2 \sin p_{\mathbf{u}}(z) \right). \end{aligned} \tag{4.31}$$

The differentials $\omega_k(u) = \partial_{N_k} p_{\mathbf{u}}(u) du$ are analytic on the hyper-elliptic curve (4.7) and are obtained as a solution of the relations (4.10).

5 3-point functions of trace operators in $\mathcal{N} = 4$ SYM

5.1 The three-point function

In a $su(2)$ sector of the SYM theory, the operators are made of two complex scalars. We consider, as in [9–11], the correlation function of three single-trace operators of the type

$$\begin{aligned} \mathcal{O}_1 &\sim \mathrm{Tr}[Z^{L_1-N_1}X^{N_1} + \dots], \\ \mathcal{O}_2 &\sim \mathrm{Tr}[\bar{Z}^{L_2-N_2}\bar{X}^{N_2} + \dots], \\ \mathcal{O}_3 &\sim \mathrm{Tr}[Z^{L_3-N_3}\bar{X}^{N_3} + \dots], \end{aligned} \tag{5.1}$$

where the omitted terms are weighted products of the same constituents taken in different order. The weights are chosen so that the operator $\mathcal{O}_i$ is an eigenstates of the dilatation operator with dimensions Δ_i. The two-point and the three-point functions are determined, up to multiplicative factors, by the conformal invariance of the theory,

$$\langle\, \mathcal{O}_i(x_i)\mathcal{O}_j(x_j)\,\rangle = L_i\, \delta_{i,j}\, \frac{\mathcal{N}_i}{|x_1 - x_2|^{2\Delta_i}}, \tag{5.2}$$

$$\langle\, \mathcal{O}_1(x_1)\mathcal{O}_2(x_2)\mathcal{O}_3(x_3)\,\rangle = \frac{L_1L_2L_3\, \sqrt{\mathcal{N}_1\mathcal{N}_2\mathcal{N}_3}\quad C_{123}(\lambda)}{|x_{12}|^{\Delta_1+\Delta_2-\Delta_3}|x_{23}|^{\Delta_2+\Delta_3-\Delta_1}|x_{31}|^{\Delta_3+\Delta_1-\Delta_2}}, \tag{5.3}$$

where $\mathcal{N}_i$ are arbitrary normalization factors.[7] The factor L_i count for the cyclic rotations of the trace operator $\mathcal{O}_i$. The structure constant $C_{123}(\lambda)$ has perturbative expansion

$$N_c\, C_{123}(\lambda) = C_{123}^{(0)} + \lambda\, C_{123}^{(1)} + \dots\,, \tag{5.4}$$

where N_c is the number of colors and λ is the 't Hooft coupling.

5.2 The structure constant in terms of scalar products of Bethe states

At tree level, the structure coefficient is a sum over all possible ways to perform the Wick contractions between the fundamental fields and their conjugates. A non-zero result is obtained only if

$$N_1 = N_2 + N_3 \tag{5.5}$$

and the number of contractions $L_{ij} = \frac{1}{2}(L_i - L_j - L_k)$ between the operators $\mathcal{O}_i$ and $\mathcal{O}_j$ are

$$L_{12} = L_1 - N_3, \quad L_{13} = N_3, \quad L_{23} = L_3 - N_3. \tag{5.6}$$

The product of all free propagators in the contractions between $\mathcal{O}_i$ and $\mathcal{O}_j$ reproduce the factor $|x_{ij}|^{-\Delta_i-\Delta_j+\Delta_k}$ in (5.3), with tree-level conformal dimensions $\Delta_i = \Delta_i^{(0)} = L_i$. By planarity, all Z fields in $\mathcal{O}_3$ must contract with $\bar{Z}$ fields in $\mathcal{O}_2$ and all $\bar{X}$ fields in $\mathcal{O}_3$ must contract with X fields in $\mathcal{O}_1$, as shown in Fig. 5. Hence there is a single term in the sum in Eq. (5.1), $\mathrm{Tr}(Z^{L_{23}}\bar{X}^{N_3})$, for which the planar contractions with $\mathcal{O}_1$ and $\mathcal{O}_3$ do not vanish.

In order to compute the tree-level structure constant $C_{123}^{(0)}$ by the method of [9], one should know the wave functions at one loop. At one loop level, the operator $\mathcal{O}_i$ is represented by a N_i-magnon

[7] Our normalization is slightly different that the normalization used in [9], namely $\mathcal{N}_i^{\text{here}} = \mathcal{N}_i^{\text{there}}/L_i$ and $C_{123}^{\text{here}} = C_{123}^{\text{there}}/\sqrt{L_1L_2L_3}$.

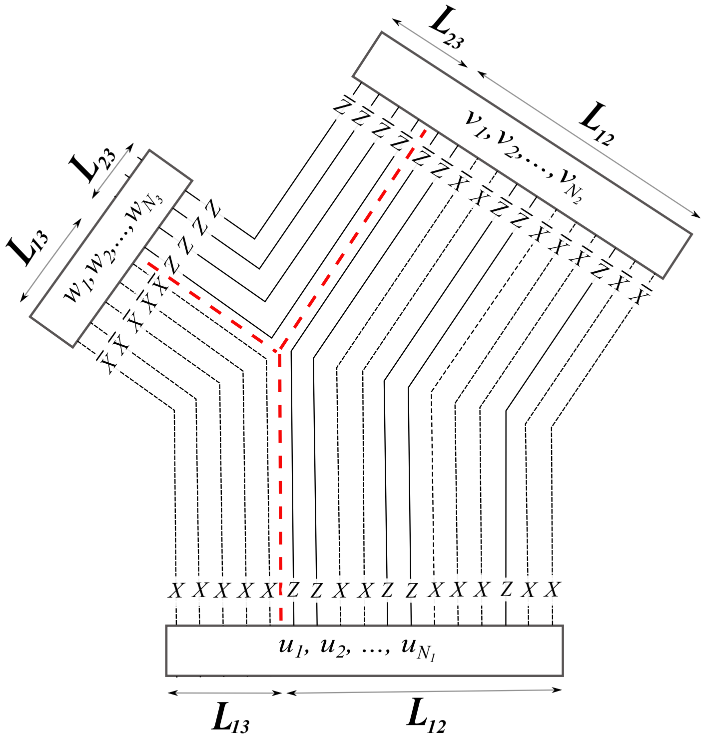

Figure 5: An example of a planar set of contractions contributing to the three-point function of the operators $\mathcal{O}_1$, $\mathcal{O}_2$ and $\mathcal{O}_3$. The $Z - \bar{Z}$ and the $X - \bar{X}$ propagators are represented respectively by continuous and dashed lines.

Bethe eigenstate[8] with energy Δ_i of the periodic $\mathrm{XXX}_{1/2}$ spin chain of length L_i $(i = 1, 2, 3)$. Let us choose the pseudovacua in the three chains as

$$\mathrm{Tr}(Z^{L_1}) \;\rightarrow |\uparrow^{L_1}\rangle\,,\; \mathrm{Tr}(\bar{Z}^{L_2}) \;\rightarrow |\uparrow^{L_2}\rangle\,,\; \mathrm{Tr}(Z^{L_3}) \;\rightarrow |\uparrow^{L_3}\rangle\,. \tag{5.7}$$

Then the three operators are determined by three Bethe eigenstates

$$\mathcal{O}_1 \rightarrow |\mathbf{u}\rangle_{L_1}, \quad \mathcal{O}_2 \rightarrow |\mathbf{v}\rangle_{L_2}, \quad \mathcal{O}_3 \rightarrow |\mathbf{w}\rangle_{L_3}, \tag{5.8}$$

with $\mathbf{u} = \{u_1, \ldots, u_{N_1}\}$, $\mathbf{v} = \{v_1, \ldots, v_{N_2}\}$, $\mathbf{w} = \{w_1, \ldots, w_{N_3}\}$. A natural normalization of the two-point function is given by choosing

$$\mathcal{N}_1 = \langle \mathbf{u}|\mathbf{u}\rangle, \quad \mathcal{N}_2 = \langle \mathbf{v}|\mathbf{v}\rangle, \quad \mathcal{N}_3 = \langle \mathbf{w}|\mathbf{w}\rangle. \tag{5.9}$$

The tailoring (= cutting+flipping+sewing) prescription of [9] gives

$$C_{123}^{(0)} = \frac{\langle \mathbf{v}\,|\mathbf{u}\rangle_{L_{12}}\; \langle\downarrow^{N_3}\;|\mathbf{w}\rangle_{L_{13}}}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle_{L_1}\langle \mathbf{v}|\mathbf{v}\rangle_{L_2}\langle \mathbf{w}|\mathbf{w}\rangle_{L_3}}}. \tag{5.10}$$

Since the inner products in the numerator are evaluated for subchains, the Bethe vectors are no more on shell. However, as proposed in Ref. [30], one can replace $\langle \mathbf{v}|\mathbf{u}\rangle_{L_{12}}$ by an inner product of the type $\langle \ldots |\mathbf{u}\rangle_{L_1}$, which can be evaluated by the Slavnov formula. This can be done after deforming the problem by introducing impurities with rapidities $\boldsymbol{\theta}^{(i)} = \{\theta_l^{(i)}\}_{l=1}^{L_i}$ at the sites of the i-th spin chain

[8] For simplicity consider highest-weight states. The generalization to arbitrary states is outlined in section 3.5.

$(i = 1, 2, 3)$. We denote the rapidities associated with the contractions between the operators $\mathcal{O}_i$ and $\mathcal{O}_j$ by $\boldsymbol{\theta}^{(ij)}$, so that $\boldsymbol{\theta}^{(1)} = \boldsymbol{\theta}^{(12)} \cup \boldsymbol{\theta}^{(13)}$, etc. Along the lines of [30] we obtain for the structure constant, up to a phase factor,

$$C_{123}^{(0)} = \frac{\langle \mathbf{v} \cup \mathbf{z} \,|\mathbf{u}\rangle_{\boldsymbol{\theta}^{(1)}} \ \langle \mathbf{z} \,|\, \mathbf{w}\rangle_{\boldsymbol{\theta}^{(3)}}}{\sqrt{\langle \mathbf{u}|\mathbf{u}\rangle_{\boldsymbol{\theta}^{(1)}} \langle \mathbf{v}|\mathbf{v}\rangle_{\boldsymbol{\theta}^{(2)}} \langle \mathbf{w}|\mathbf{w}\rangle_{\boldsymbol{\theta}^{(3)}}}} \qquad \text{with} \quad \mathbf{z} = \boldsymbol{\theta}^{(13)} + i/2. \tag{5.11}$$

In this way we expressed the structure constant in terms of the quantities evaluated in Sections 3.5 and 3.6.

5.3 The BPS limit

The structure coefficient should be normalized so that in the limit when all rapidities go to infinity it tends to the structure coefficient for three BPS fields [55],

$$\lim_{\mathbf{u},\mathbf{v},\mathbf{w}\to\infty} C_{123}^{(0)} = C_{123}^{\mathrm{BPS}}. \tag{5.12}$$

From (3.34) we obtain the expected result

$$C_{123}^{\mathrm{BPS}} = \frac{N_1!(N_1-N_3)! \binom{L_1-N_3}{N_2} \times N_3! \binom{L_3-N_3}{N_3}}{\sqrt{(N_1!)^2 \binom{L_1}{N_1} \times (N_2!)^2 \binom{L_2}{N_2} \times (N_3!)^2 \binom{L_3}{N_3}}} = \frac{\binom{L_{12}}{N_2} \binom{L_{23}}{N_3}}{\sqrt{\binom{L_1}{N_1} \binom{L_2}{N_2} \binom{L_3}{N_3}}}. \tag{5.13}$$

5.4 The limit of three classical operators

Here we take the limit when the three fields become classical,

$$L_i \to \infty \quad \text{with} \quad \alpha_i = \frac{N_i}{L_i} \quad \text{fixed} \qquad (i = 1, 2, 3). \tag{5.14}$$

using the results of Section 4. The two factors in the numerator in (5.11) are evaluated using (4.29),

$$\begin{aligned} \log\langle \mathbf{v} \cup \mathbf{z} \,|\mathbf{u}\rangle_{\boldsymbol{\theta}^{(1)}} &= i\pi N_1 + \oint_{A_{\mathbf{u}} \cup A_{\mathbf{v}}} \frac{du}{2\pi} \, \mathrm{Li}_2 \left(\exp i[G_{\mathbf{u}} + G_{\mathbf{v}} - G_{\boldsymbol{\theta}^{(12)}}]\right) & (5.15) \\ &= i\pi N_1 + \oint_{A_{\mathbf{u}} \cup A_{\mathbf{v}}} \frac{du}{2\pi} \, \mathrm{Li}_2 \left(\exp i[p_{\mathbf{u}} + p_{\mathbf{v}} + \tfrac{1}{2} G_{\boldsymbol{\theta}^{(3)}}]\right) , \\ \log\langle \mathbf{z} \,|\, \mathbf{w}\rangle_{\boldsymbol{\theta}^{(3)}} &= i\pi N_3 + \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \, \mathrm{Li}_2 (\exp i[G_{\mathbf{w}} - G_{\boldsymbol{\theta}^{(23)}}]) & (5.16) \\ &= i\pi N_3 + \oint_{A_{\mathbf{w}}} \frac{du}{2\pi} \, \mathrm{Li}_2 \left(\exp i[p_{\mathbf{w}} + \tfrac{1}{2} G_{\boldsymbol{\theta}^{(2)}} - \tfrac{1}{2} G_{\boldsymbol{\theta}^{(1)}}]\right) , \end{aligned}$$

where we introduced the three pseudo-momenta (for $\phi = 0$)

$$p_{\mathbf{u}} = G_{\mathbf{u}} - \tfrac{1}{2} G_{\boldsymbol{\theta}^{(1)}}, \quad p_{\mathbf{v}} = G_{\mathbf{v}} - \tfrac{1}{2} G_{\boldsymbol{\theta}^{(2)}}, \quad p_{\mathbf{w}} = G_{\mathbf{w}} - \tfrac{1}{2} G_{\boldsymbol{\theta}^{(3)}}. \tag{5.17}$$

The norms in the denominator are evaluated using (4.27). Collecting all terms we find

$$\begin{aligned}\log C_{123}^{(0)} &\simeq \oint_{A_{\mathbf{u}}\cup A_{\mathbf{v}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{i[p_{\mathbf{u}}+p_{\mathbf{v}}+\frac{1}{2}G_{\boldsymbol{\theta}(3)}]}\big) + \oint_{A_{\mathbf{w}}} \frac{du}{2\pi}\,\mathrm{Li}_2\Big(e^{i[p_{\mathbf{w}}+\frac{1}{2}G_{\boldsymbol{\theta}(2)}-\frac{1}{2}G_{\boldsymbol{\theta}(1)}]}\Big) \\ &- \tfrac{1}{2}\int_{A_{\mathbf{u}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{2ip_{\mathbf{u}}}\big) - \tfrac{1}{2}\int_{A_{\mathbf{v}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{2ip_{\mathbf{v}}}\big) - \tfrac{1}{2}\int_{A_{\mathbf{w}}} \frac{dz}{2\pi}\,\mathrm{Li}_2\big(e^{2ip_{\mathbf{w}}}\big). \end{aligned} \tag{5.18}$$

The tree-level structure constant is obtained by setting all inhomogeneity parameters to zero:

$$\begin{aligned}\log C_{123}^{(0)} \simeq & \oint_{A_{\mathbf{u}}\cup A_{\mathbf{v}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{ip_{\mathbf{u}}(u)+ip_{\mathbf{v}}(u)+iL_3/2u}\big) + \oint_{A_{\mathbf{w}}} \frac{du}{2\pi}\,\mathrm{Li}_2\Big(e^{ip_{\mathbf{w}}(u)+i(L_2-L_1)/2u}\Big) \\ & - \tfrac{1}{2}\int_{A_{\mathbf{u}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{2ip_{\mathbf{u}}(u)}\big) - \tfrac{1}{2}\int_{A_{\mathbf{v}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{2ip_{\mathbf{v}}(u)}\big) - \tfrac{1}{2}\int_{A_{\mathbf{w}}} \frac{du}{2\pi}\,\mathrm{Li}_2\big(e^{2ip_{\mathbf{w}}(u)}\big). \end{aligned} \tag{5.19}$$

To make connection with the result of [11], one should send all u's to infinity, which is the same as taking $G_{\mathbf{u}} = 0$ and neglecting the integration along $A_{\mathbf{u}}$.

6 Conclusions and speculations

The principal result of this work is the operator factorization formula (3.17) for the scalar product and its clasicial limit (4.11)-(4.2). Using this result, we were able to write a compact expression for the correlation function of three non-BPS operators in maximally supersymmetric Yang-Mills theory in the classical limit when the traces become large. Our starting point was Foda's determinant expression for the three-point structure constant [30]. The determinant formula of [30] was derived supposing that the three operators are deformed by a set of inhomogeneity parameters, whose values can be chosen at will. We computed the bosonized determinant expression for inhomogeneous problem and took, as in [30], the homogeneous limit at the very end in order to avoid spurious singularities.

Another reason to treat the inhomogeneous problem is that this allows to extend, as argued in [56, 57], the tree-level result to higher orders in λ. Gromov and Vieira [56] showed that knowing the tree level solution for $C_{123}^{(0)}$ in presence of impurities, one can obtain the one-loop and the two-loop corrections by applying a special differential operator acting on the inhomogeneity parameters $\boldsymbol{\theta}^{(1)}$, $\boldsymbol{\theta}^{(2)}$ and $\boldsymbol{\theta}^{(3)}$. Serban [57] proposed that this statement can be extended to all loops in the BDS model [58], i.e. when the dressing phase is not taken into account, and in the limit of large lengths L_1, L_2, L_3. The analysis of [57] leads to the prescription that the higher loop corrections can be taken into account only by modifying the pseudo-momenta. For example, the three-loop result for the structure constant is obtained by changing the pseudo-momenta $p_{\mathbf{u}}, p_{\mathbf{v}}$ and $p_{\mathbf{w}}$ according to the three-loop Bethe ansatz equations [58].

According to [57], for finite value of the 't Hooft coupling λ, the structure constant for three classical operators in the BDS model is given by (5.18) with a particular distribution of the L_i inhomogeneity parameters[9] in the interval $[-2g, 2g]$, where $\lambda = 16\pi^2 g^2$. For this distribution the resolvents for the inhomogeneities associated with the three chains are given by

$$G_{\boldsymbol{\theta}^{(i)}}(u) = \frac{L_i}{\sqrt{u^2-4g^2}} = \frac{L_i}{x}\frac{dx}{du} \qquad (i = 1, 2, 3), \tag{6.1}$$

[9] This is the distribution of L_i equal charges confined to the segment $[-2g, 2g]$ in absence of external potential.

where x is the 'Zhukovsky variable' defined by

$$u = x + g^2/x. \tag{6.2}$$

This has the same effect as changing the vacuum eigenvalues $a(u)$ and $d(u)$ of the transfer matrix, eq. (2.11), to

$$a(u) = [x(u + \tfrac{i}{2})]^L, \;\; d(u) = [x(u - \tfrac{i}{2})]^L. \tag{6.3}$$

In the strong coupling regime it is convenient to perform the change the variable (6.2) in the contour integrals in (5.18). We should mention here that the replacement (6.3) as a possible way to take into account (some of) the loop corrections was previously discussed in [10].

Assuming that this conjecture is correct, the expression (5.18) with the choice (6.1) for the inhomoheneities will give the all-loops result for the structure constant for the BDS model in which the dressing phase is neglected. On the other hand, the effect of the dressing phase [59], in the limit of large L_1, L_2, L_3, is that the pseudo-momentum is modified as

$$p(u) \to p^{\text{BES}}(u) = p(u) - i \log \sigma^{\text{BES}}(u). \tag{6.4}$$

The fact that (5.18) depends only on the three quasi-momenta $p_{\mathbf{u}}, p_{\mathbf{v}}, p_{\mathbf{w}}$ and the resolvents $G_{\boldsymbol{\theta}^{(i)}} = L_i \partial_u \log x$ invites one to consider the possibility that the result at finite 't Hooft coupling λ is given in the classical limit again by (5.18), with the three pseudo-momenta modified according to (6.4).

It is natural to expect that a systematic approach to the correlation functions of heavy operators in the full field strength multiplet of $\mathcal{N} = 4$ SYM should be some extension of the algebraic curve method used in the spectral problem in [12–14].[10] In the case of three long-trace operators, the structure constant C_{123} is expected to be described by the ensemble of three algebraic curves, associated with the pseudo-momenta $p_{\mathbf{u}}, p_{\mathbf{v}}, p_{\mathbf{w}}$. In order to build a general algebraic curve formalism, one should learn how to compute more general inner products, at least in the classical limit. An interesting development in this direction was reported by Wheeler [61], who wrote determinant formulas analogous to (3.25) are obtained for the generalized model with $SU(3)$ symmetry.

Finally, it is certainly worth to try to adjust the expression (5.18) for the non-compact rank-one sectors of the full symmetry $PSU(2,2|4)$, such as $SL(2,\mathbb{R})$ and $SU(1,1)$, where the integration contours should be placed along the real axis.

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